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SPHERICAL SPACE FORMS WITH NORMAL CONTACT METRIC 3-STRUCTURE

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Introduction

The theory of contact structure was initiated by S. S. Chern [2] in 1953 in studying pseudo-groups and was developed further by W. M. Boothby & H. C. Wang [1]. G. Reeb [7] also gave an important contribution a little earlier than them to the study of dynamical systems. The generalization to almost contact structure was first studied by J. W. Gray [3] in 1959. The present author [8], [10], [11] with Y. Hatakeyama introduced a new way to study these structures in 1960 by initiating the notions of (almost) contact structure, (almost) contact metric structure, torsion tensor and normality of the structure. Since then many papers on (almost) contact (metric) structures and related topics have been published by many authors.

Recently, Y. Y. Kuo [6] studied Riemannian manifolds with a (almost) contact 3-structure and gave some fundamental properties. Then, S. Tachibana and W. N. Yu [12], S. Tanno [13] and T. Kashiwada [4] studied Riemannian manifolds with a normal contact 3-structure. The purpose of this paper is to study spherical space forms which admit a normal contact metric 3-structure. For the notations on contact structures we refer to the paper [9].

1. Quaternion structure in E^{4d}

1.1. First, let us consider the 4-dimensional case. Let

$$x = x_0 + x_1 i + x_2 j + x_3 k \tag{1.1}$$

be an element of the quaternion algebra Q where x_0, x_1, x_2, x_3 belong to the field of real numbers. We identify x with the vector of components (x_0, x_1, x_2, x_3) of a Euclidean vector space E^4 with respect to an orthonormal basis. Now consider three linear mappings I, J, K of E^4 onto itself defined by

$$Ix = -xi \, , \quad Jx = -xj \, , \quad Kx = -xk \, . \tag{1.2}$$

Then they are complex structures in E^4 and satisfy the relations

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$$(1.3)\qquad \begin{aligned} &I^2 = J^2 = K^2 = -1\ , \qquad && JK = -KJ = I\ ,\\ &KI = -IK = J\ , && IJ = -JI = K\ , \end{aligned}$$

where 1 on the right hand side means the identity mapping. All of the complex structures I, J, K are Hermitian with respect to the ordinary Euclidean metric of E^4. If we express the mappings I, J and K as linear transformations of components of a vector, then they are represented by the following matrices:

$$(1.4)\qquad \begin{aligned} &I = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},\\ &K = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

In general, three complex structures F_1, F_2, F_3 in a vector space V^4 over the real number field are said to define a *quaternion structure* if they satisfy the relations

$$(1.5)\qquad F_\lambda^2 = -1\ , \qquad F_\lambda F_\mu = -F_\mu F_\lambda = F_\nu\ ,$$

where (λ, μ, ν) means any even permutation of $1, 2, 3$. In this case, there exists a Euclidean metric g_0 in V^4, with respect to which all the complex structures F_1, F_2, F_3 are Hermitian, so that we may consider V^4 as E^4. We take a unit vector e_1 and put

$$(1.6)\qquad e_2 = -F_1 e_1\ , \quad e_3 = -F_2 e_1\ , \quad e_4 = -F_3 e_1\ .$$

Then it is easy to verify that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of V^4, with respect to which F_1, F_2, F_3, g_0 are represented by matrices in (1.4) and the unit matrix. Hence there exists essentially only one quaternion structure for V^4.

1.2. In the same way as above we can define a quaternion structure in a vector space V^{4d} $(d > 0)$ over the real number field by three linear mappings F_1, F_2, F_3 of V^{4d} onto itself, which satisfy (1.5). Then by taking an orthonormal basis of V^{4d} with respect to a Euclidean metric g_0 which is Hermitian for all of F_1, F_2, F_3, we can express the complex structure F_1 (resp. F_2, F_3) and the metric g_0 by the diagonal matrix of order d with the matrix I (resp. J, K) of order 4 and the unit matrix E of order 4 respectively as each entry on the principal diagonal. Thus F_1, F_2, F_3 and g_0 are reduced to

$$(1.7)\qquad \begin{aligned} &I_d = I \times I \times \cdots \times I\ , \qquad && J_d = J \times J \times \cdots \times J\ ,\\ &K_d = K \times K \times \cdots \times K\ , && g_d = E \times E \times \cdots \times E\ , \end{aligned}$$

so that we obtain

Lemma 1. *In V^{4d}, there exists essentially only one quaternion structure.*

2. Normal contact metric structure on S^{4d-1}

Let

$$S^{4d-1} = \{x \mid x \in E^{4d},\ \|x\| = 1\}$$

be a unit hypersphere in a Euclidean vector space E^{4d}. If we put

$$\xi_x = I_d x\ , \qquad x \in S^{4d-1}\ , \tag{2.1}$$

then ξ defines a unit vector field on S^{4d-1}. We denote by g the metric on S^{4d-1} induced from the Euclidean metric g_0 of E^{4d} by the inclusion map of S^{4d-1} into E^{4d}, and by π the natural orthogonal projection of the tangent spaces

$$T_x(E^{4d}) \to T_x(S^{4d-1})\ , \qquad x \in S^{4d-1}\ .$$

For η and Φ defined by

$$\eta(X) = g(\xi, X)\ , \qquad \Phi X = -\pi \cdot I_d X (= -I_d X - \eta(X)x)\ , \tag{2.2}$$

we have

$$\eta(\xi) = 1\ , \qquad \Phi \cdot \Phi X = -X + \eta(X)\xi\ , \tag{2.3}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)\ , \tag{2.4}$$

$$d\eta(X, Y) = g(X, \Phi Y)\ , \tag{2.5}$$

$$N(X, Y) = 0 \qquad \text{(normality)}\ , \tag{2.6}$$

where X, Y are arbitrary vector fields on S^{4d-1} and

$$\begin{aligned} N(X, Y) = [X, Y] &+ \Phi[\Phi X, Y] + \Phi[X, \Phi Y] \\ &- [\Phi X, \Phi Y] - \{X \cdot \eta(Y) - Y \cdot \eta(X)\}\xi\ . \end{aligned} \tag{2.7}$$

In other words, (Φ, ξ, η, g) gives a normal contact metric structure on S^{4d-1}.

Conversely, a set of tensor fields (Φ, ξ, η, g) over S^{4d-1}, where g is the metric induced from the Euclidean metric g_0 in E^{4d}, is said to be a normal contact metric structure on S^{4d-1} if it satisfies $(2.2)_1$ and (2.3)–(2.6).

Lemma 2. *Let (Φ, ξ, η, g) be a normal contact metric structure on S^{4d-1}. Then there exists a complex structure F in E^{4d} such that*

$$\xi = Fx\ , \quad x \in S^{4d-1}\ , \quad \Phi X = -\pi F X (= -FX - \eta(X)x) \tag{2.8}$$

hold.

Proof. Each point of $E^{4d} - \{0\}$ (0 being the center of S^{4d-1}) is represented uniquely by its polar coordinates (x, r), $x \in S^{4d-1}$, $r > 0$. Define a linear mapping F of the space of vectors at (x, r) onto itself by

$$FX = -\Phi X - \eta(X)x\ , \qquad Fx = \xi\ , \tag{2.9}$$

where X is an arbitrary vector orthogonal to x. Then we can easily verify that F is an almost complex structure in $E^{4d} - \{0\}$. Furthermore, (2.9) implies that F is constant along each open ray through 0. We shall show that F is also constant on S^{4d-1}. To this end, we notice the derived equations of Gauss and Weingarten for S^{4d-1}:

$$\bar{\nabla}_X Y = \nabla_X Y - g(X, Y)x\ , \qquad \bar{\nabla}_X x = X\ , \tag{2.10}$$

where X, Y are tangent vector fields to S^{4d-1} in a coordinate neighborhood U of S^{4d-1}, and $\bar{\nabla}_X$, ∇_X are covariant derivatives with respect to the standard Riemannian metrics of E^{4d} and S^{4d-1} respectively. By virtue of the relations (2.10) and

$$(\nabla_X \Phi)Y = g(X, Y)\xi - \eta(Y)X\ , \qquad (\nabla_X \eta)(Y) = -g(\Phi X, Y) \tag{2.11}$$

(cf. [9], [11]), the left hand side of

$$\bar{\nabla}_X(FY) = (\bar{\nabla}_X F)Y + F\bar{\nabla}_X Y$$

is transformed to

$$\begin{aligned} -\bar{\nabla}_X\{\Phi Y + \eta(Y)x\} &= -\{\nabla_X(\Phi Y) - g(X, \Phi Y)x\} - \nabla_X(\eta(Y))x - \eta(Y)X \\ &= -\{g(X, Y)\xi + \Phi\nabla_X Y + \eta(\nabla_X Y)x\}\ , \end{aligned}$$

and the right hand side to

$$(\bar{\nabla}_X F)Y - \{\Phi\nabla_X Y + \eta(\nabla_X Y)x + g(X, Y)\xi\}\ .$$

Thus

$$(\bar{\nabla}_X F)Y = 0\ . \tag{2.12}$$

Similarly, by transforming both sides of

$$\bar{\nabla}_X(Fx) = (\bar{\nabla}_X F)x + F\bar{\nabla}_X x$$

by virtue of (2.10) and

$$\nabla_X \xi = -\Phi X \tag{2.13}$$

(cf. [9], [11]), we get

(2.14) $$(\bar{\nabla}_X F)x = 0 \;,$$

which together with (2.12) implies that $\bar{\nabla}_X F = 0$, so that F is constant in U. Therefore F is constant in $U \times (0, \infty)$. Now, it is easy to see that F is constant in $E^{4d} - \{0\}$ so that we can extend F to E^{4d} differentiably. Hence F is a complex structure in E^{4d}.

3. Normal contact metric 3-structure on S^{4d-1}

3.1. Let F_1, F_2, F_3 be a quaternion structure in E^{4d}, and for each λ, $\lambda = 1, 2, 3$, put

(3.1) $$\xi_\lambda = F_\lambda x \;, \qquad x \in S^{4d-1} \;,$$

(3.2) $$\eta_\lambda(X) = g(\xi_\lambda, X) \;, \qquad \Phi_\lambda(X) = -F_\lambda X - \eta_\lambda(X)x \;.$$

Then $(\Phi_\lambda, \xi_\lambda, \eta_\lambda, g)$'s define three normal contact metric structures on S^{4d-1} with the same Riemannian metric g. Since F_λ, for $\lambda = 1, 2, 3$, define a quaternion structure in E^{4d}, we see that $\Phi_\lambda, \xi_\lambda, \eta_\lambda$ satisfy

(3.3) $$\begin{aligned} &\Phi_\nu = -\Phi_\lambda \Phi_\mu + \xi_\lambda \otimes \eta_\mu = \Phi_\mu \Phi_\lambda - \xi_\mu \otimes \eta_\lambda \;, \\ &\xi_\nu = -\Phi_\lambda \xi_\mu = \Phi_\mu \xi_\lambda \;, \qquad \eta_\nu = -\eta_\lambda \Phi_\mu = \eta_\mu \Phi_\lambda \;, \end{aligned}$$

where (λ, μ, ν) is an even permutation of $(1, 2, 3)$. (3.3) implies

(3.4) $$\eta_\lambda(\xi_\mu) = 0 \qquad \text{if} \quad \lambda \neq \mu \;.$$

In general, three normal contact metric structures on S^{4d-1} satisfying (3.3) are said to define a normal contact metric 3-structure on S^{4d-1}. Lemma 2 and the above argument show that this structure corresponds to a quaternion structure in E^{4d}. As each quaternion structure is transformed to the standard quaternion structure by § 1.2, we can identify F_1, F_2, F_3 with I_d, J_d, K_d without any loss of generality.

3.2. Now let us consider the antipodal map T of S^{4d-1} onto itself. Then from

(3.5) $$TF_\lambda = F_\lambda T \;,$$

it follows that

(3.6) $$\xi_{\lambda, Tx} = T\xi_{\lambda, x} \;, \qquad x \in S^{4d-1} \;,$$

and that

$$\eta_\lambda(TX) = g(\xi_{\lambda, Tx}, TX) = g(T\xi_{\lambda, x}, TX) = g(\xi_{\lambda, x}, X) \;,$$

$$\eta_\lambda(TX) = \eta_\lambda(X) \tag{3.7}$$

for $X \in T_x(S^{4d-1})$. We have also

$$\Phi_\lambda TX = T\Phi_\lambda X , \tag{3.8}$$

because

$$\Phi_\lambda TX = -F_\lambda TX - \eta_\lambda(TX)Tx = -TF_\lambda X - \eta_\lambda(X)Tx = T\Phi_\lambda X .$$

Thus each of $(\Phi_\lambda, \xi_\lambda, \eta_\lambda)$'s is invariant under T, and, of course, g is invariant under T too. Hence $(\Phi_\lambda, \xi_\lambda, \eta_\lambda, g)$, for $\lambda = 1, 2, 3$, define a normal contact metric 3-structure in the projective space P^{4d-1} with the natural Riemannian metric induced from S^{4d-1}.

3.3. Taking account of the fact that S^{4d-1} and P^{4d-1} are spherical space forms, it is natural to consider the problem of determining all spherical space forms with a normal contact metric 3-structure.

Now each spherical space form of dimension $4d-1$ is isometric to a certain space of orbits S^{4d-1}/Γ, where Γ is a finite fixed point free subgroup of $O(4d)$, the isometry group of S^{4d-1}. So we may formulate our problem as follows: Determine all distinct (i.e., nonisometric) spherical space forms S^{4d-1}/Γ such that Γ leaves invariant each of the three normal contact metric structures $(\Phi_\lambda, \xi_\lambda, \eta_\lambda)$.

Since two distinct finite fixed point free subgroups Γ and Γ' of $O(4d)$ in general do not give nonisometric space forms, the criterion that they give isometric space forms is given as follows. Noticing that Γ (resp. Γ') can be regarded as a direct sum of several fixed point free irreducible orthogonal representations $\sigma_1, \cdots, \sigma_p$ (resp. $\sigma'_1, \cdots, \sigma'_q$) of an abstract group G $(\cong\Gamma)$(resp. G' $(\cong\Gamma')$), we say that Γ and Γ' are equivalent if and only if (i) $G \cong G'$, (ii) $p = q$ and (iii) there exist a permutation π of $(1, \cdots, p)$ and an isomorphism α of G onto G' such that $\sigma'_\lambda \circ \alpha$ and $\sigma_{\pi(\lambda)}$ are conjugate. Then two spherical space forms $O(4d)/\Gamma$ and $O(4d)/\Gamma'$ are isometric if and only if Γ and Γ' are equivalent (cf. G. Vincent [14], J. A. Wolf [15]).

On the other hand, (3.6) for $T \in \Gamma$ implies (3.5), and (3.5) for $T \in \Gamma$ implies (3.6)–(3.8). Thus the problem of finding all distinct spherical space forms with a normal contact metric 3-structure reduces to the one of finding a representative from each equivalence class of finite fixed point free subgroups of $O(4d)$, which leave I_d, J_d and K_d invariant. Any such representative Γ gives a required spherical space form S^{4d-1}/Γ.

4. Determination of all spherical space forms with a normal contact metric 3-structure

4.1. First let us study the simplest case where $d = 1$, i.e., the case of S^3, in which (3.5) can be written as

$$IT = TI\ ,\quad JT = TJ\ ,\quad KT = TK\ . \tag{4.1}$$

The linear mapping T is expressed as a linear transformation of components of a vector by a matrix $(t_{\alpha\beta})$, $\alpha, \beta = 1, \cdots, 4$. Noticing that I, J and K are given by matrices in (1.4), we therefore see that $(4.1)_1$ and $(4.1)_2$ reduce to

$$\begin{aligned} &t_{11} = t_{22}\ , && t_{33} = t_{44}\ , && t_{12} = -t_{21}\ , && t_{34} = -t_{43}\ , \\ &t_{13} = -t_{24}\ , && t_{14} = t_{23}\ , && t_{31} = -t_{42}\ , && t_{41} = t_{32} \end{aligned}$$

and

$$\begin{aligned} &t_{11} = t_{33}\ , && t_{22} = t_{44}\ , && t_{12} = t_{34}\ , && t_{21} = t_{43}\ , \\ &t_{13} = -t_{31}\ , && t_{24} = -t_{42}\ , && t_{14} = -t_{32}\ , && t_{23} = -t_{41} \end{aligned}$$

respectively. (We do not write out similar equations corresponding to $(4.1)_3$, since $(4.1)_3$ is a consequence of $(4.1)_1$ and $(4.1)_2$ together with $K = IJ$.) From these relations it follows that T has the form

$$T = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} . \tag{4.2}$$

As $T \in O(4)$, $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. If we express a point x of S^3 by a quaternion as

$$x = x_0 + x_1 i + x_2 j + x_3 k\ , \qquad \|x\| = 1\ ,$$

and consider also the quaternion

$$a = a_0 + a_1 i + a_2 j + a_3 k\ , \qquad \|a\| = 1\ , \tag{4.3}$$

then we can casily verify that

$$Tx = ax\ . \tag{4.4}$$

Thus the mapping T of $O(4)$ leaves the complex structures I, J invariant, and K is a left translation of the multiplicative group Q' of the unit quaternions.

The mapping of S^3 (resp. P^3) onto itself induced by such T of $O(4)$ is called a Clifford translation of the second kind of S^3 (resp. P^3) (cf. F. Klein [5]), and we may easily see that it is fixed point free. So in the case of S^3 our problem reduces to the following one: Determine a representative from each equivalence class of the finite subgroups of the group of Clifford translations. (We omit the adjective "of the second kind" for brevity, as by Lemma 1 it is not necessary to consider Clifford translations of the first kind corresponding to the right translations $x' = xa$.)

4.2. The answer to the problem is well known. To explain it we need the definitions of some groups. Define the homomorphism $\tau\colon Q' \to SO(4)$ by

$$\tau(a)(x) = axa^{-1}\,, \qquad a \in Q',\ x \in Q\,, \tag{4.5}$$

and consider the groups

$$D_m^* = \tau^{-1}(D_m)\,, \quad T^* = \tau^{-1}(T)\,, \quad O^* = \tau^{-1}(O)\,, \quad I^* = \tau^{-1}(I)\,,$$

where D_m is the dihedral group (i.e., the group of rotations in E^3 of a regular m-sided polygon in a plane), and T, O and I are respectively the tetrahedral, octahedral and icosahedral groups (i.e., the groups of rotations of a regular tetrahedron, regular octahedron and regular icosahedron). They are respectively called the binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral groups.

4.3. All finite subgroups of Clifford translations on S^3 are then equivalent to either one of

(i) $\Gamma = \{1\}$ (1 being the identity mapping),
(ii) $\Gamma = \{\pm 1\}$,
(iii) Γ is the cyclic group of order $q > 2$ generated by

$$T = \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}, \tag{4.5}$$

where we have put $c = \cos 2\pi/q$, $s = \sin 2\pi/q$,

(iv) Γ is the group of Clifford translations which corresponds to a binary dihedral group or one of the binary polyhedral group T^*, O^* and I^*.

Hence we obtain

Theorem 1. *All 3-dimensional spherical space forms with a normal contact metric 3-structure are given as S^3/Γ, where Γ is any one of the subgroups of Clifford translations given by* (i), (ii), (iii) *and* (iv).

Remark. $\Gamma = \{1\}$ and $\{\pm 1\}$ give S^3 and P^3. S^3/Γ for Γ of type (iii) is the so-called lens space whose fundamental group π is isomorphic to Γ.

By a theorem of J. A. Wolf [15], we have, as a corollary,

Theorem 2. *A necessary and sufficient condition that a 3-dimensional spherical space form admits a normal contact metric 3-structure is that it be a homogeneous Riemannian manifold.*

4.4. In the general case where $d > 1$, the complex structures I_d, J_d, K_d have the form (1.7) so that the linear mapping T_d of E^{4d} leaving all of these structures invariant is given by a matrix of order d whose entries are matrices of the form (4.2). As $T \in O(4d)$, this means that T is an element of the real representation of the symplectic group $S_p(d)$. Thus our problem reduces to the following one: Determine a representative from each equivalence class of the finite fixed point free subgroups of $S_p(d)$. We do not treat this algebraic problem in this paper. However, examples are given by $\Gamma_d = \Gamma \times \Gamma \times \cdots \times \Gamma$ (d factors), where Γ is any one of the groups given in (i)–(iv) of § 4.3. They give S^{4d-1}, the projective space P^{4d-1}, the lens space L^{4d-1} and so on as spherical space forms with a normal contact metric 3-structure.

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